Adaptive Estimation of the Dynamic Linear Model with Fixed Effects*

By Tiemen Woutersen and Marcel Voia†

May 2002

1 Introduction

THE ANALYSIS OF THE DYNAMIC LINEAR MODEL with fixed effects has been subject of some attention in econometrics for almost two decades. The popularity of this linear model might be due to the fact that it is the simplest model in which Heckman's (1981a and 1981b) spurious correlation and state dependence can be studied. Also, in practice growth models such as the dynamic version of the Solow's (1956) are fitting very well into the class of dynamic linear models with fixed effects. Sollow's model became important once the paper of Mankiw, Romer and Weil (1992) used it to fit data on GDP, savings and labor force for 121 countries during $1960-1985$. They found then that aproximatively 80 percent of the variation in the above mentioned variables confirm model's predictions.

Refering to the model estimation, Nickell (1981) showed that the maximum likelihood suffers from the incidental parameter problem of Neyman and Scott (1948). In particular, Nickell shows that the inconsistency of the maximum likelihood estimator is $O(T^{-1})$. Econometricians have subsequently developed moment estimators. Examples include Anderson and Hsiao (1982), Holtz-Eakin, Newey and Rosen (1988), Arrelano and Bond (1991), and Ahn and Schmidt (1995). Combining moment restrictions can be difficult, especially, when some moments become uninformative for particulr regions of the parameter space. Neway

*We gratefully acknowlegde stimulating suggestions from Jan Kieviet. The CIBC Human Capital and Productivity Program provided financial support. All errors are ours.

†Correspondence addresses: Brown University, Box B, RI, 02912, USA, and University of Western Ontario, Department of Economics, Social Science Centre, London, Ontario, N6A 5C2, Canada. Email: Tony_Lancaster@brown.edu, mvoia@uwo.ca, and twouters@uwo.ca.

and Smith (2001) also give a general discussion on how combining moments usually causes a higher order bias. The small sample bias of these moment estimators motivates the bias corrected least squares estimator of Kieviet (1995). Alvarez and Arellano (1998) developed an alternative asymptotic where the number of individuals, N, increases as well as the number of observations per individual, T. Using this alternative asymptotics, Hahn, Hausman and Kuersteiner (2001) develop an expression for the bias for the case in which $N \propto T$ then, Hahn et al. developed an estimator for the dynamic linear model without regressors. Hahn and Kuersteiner (2002) developed a bias corrected MLE which is asymptotically unbiased and efficient.

Lancaster (1997 and 2000) proposes to approximately separate the parameters of interest from the nuisance parameters. In particular, Lancaster uses a parametrization of the likelihood that has a block diagonal information matrix. That is, the cross derivatives of the log likelihood of the nuisance parameters and parameters of interest is zero in expectation. Lancaster (1997 and 2000) then integrates out all fixed effects. Woutersen (2001) shows that information-orthogonality reduces the bias of this integrated likelihood estimator to $O(T^{-2})$ and that the integrated likelihood estimator is asymptotically unbiased and adaptive if $T \propto N^{\alpha}$ where $\alpha > \frac{1}{3}$. That is, the integrated likelihood estimator is as efficient as the infeasible maximum likelihood estimator that assumes the values of the nuisance parameters to be known.

This paper presents a new adaptiveness result for the integrated likelihood estimator that integrates out individual specific effects. We show that the integrated likelihood estimator is adaptive for any asymptotic in which T increases as long as the integrated likelihood estimator is consistent for N increasing and T being fixed. This theorem implies adaptiveness of Lancaster's (1997) estimator of the dynamic linear model without regressors. We also derive an information orthogonal parametrization for the dynamic model with regressors. For this model, the integrated likelihood estimator is asymptotically unbiased and adaptive as long as $T \propto N^{\alpha}$ where $\alpha > \frac{1}{3}$.

{Lancaster (1997) gives a parametrization of the dynamic linear model that approximates

an information-orthogonal parametrization if $N \to \infty$. However, information-orthogonality is a finite sample property. Simulations show that an implicit parametrization works better; this was somewhat surprising to me (tw) since N was larger than T in all simulations}

This paper is organized as follows. Section 2 reviews the integrated likelihood estimator and information orthogonality. Section 3 derives an information orthogonal parametrization for the dynamic linear model with fixed effects and regressors. Section 4 gives adaptiveness results for the integrated likelihood estimator (case without regressors). Section 5 gives simulation results and section 6 concludes.

2 The Integrated Likelihood Estimator and Orthogonality

Suppose we observe N individuals for T periods. Let the log likelihood contribution of the t^{th} spell of individual i be denoted by L^{it}. Summing over the contribution of individual i yields the log likelihood contribution,

$$L^i(\beta, \lambda_i) = \sum_t L^{it}(\beta, \lambda_i),$$

where β is the common parameter and λ_i is the individual specific effect. Suppose that the parameter β is of interest and that the fixed effect λ_i is a nuisance parameter that controls for heterogeneity. This paper considers elimination of nuisance parameters by integration. This Bayesian treatment of nuisance parameters is straightforward: formulate a prior on all the nuisance parameters and then integrate the likelihood with respect to that prior distribution of the nuisance parameters, (see Gelman et al. (1995) for an overview). For a panel data model with fixed effects this means that we have to specify priors on the common parameters and all the fixed effects. Berger et al. (1999) review integrated likelihood methods in which flat priors are used for both the parameter of interest and the nuisance parameters. The individual specific nuisance parameters are then eliminated by integration. We denote the logarithm of the integrated likelihood contribution by $L^{i,I}$, i.e.

$$L^{i,I}(\beta) = \ln \int e^{L^i} d\lambda_i.$$

Summing over i yields the logarithm of the integrated likelihood,

$$L^I(\beta) = \sum_i L^{i,I}(\beta) = \sum_i \ln \int e^{L^i} d\lambda_i. \tag{1}$$

After integrating out the fixed effects, the mode of the integrated likelihood can be used as an estimator[1] We thus define the *integrated likelihood estimator* $\hat{\beta}$ to be the mode of $L^I(\beta)$:

$$\hat{\beta} = \arg\max_\beta L^I(\beta).$$

A parametrization of the likelihood is *information-orthogonal* if the information matrix is block diagonal. That is

$$\begin{aligned} EL_{\beta\lambda}(\beta_0, \lambda_0) &= 0 \\ &\text{i.e.} \\ \int_{t_{\min}}^{t_{\max}} L_{\beta\lambda}(\beta_0, \lambda_0) e^{L(\beta_0,\lambda_0)} dt &= 0, \end{aligned}$$

where t denotes the dependent variable, $t \in [t_{\min}, t_{\max}]$ and β_0, λ_0 denote the true value of the parameters. Cox and Reid (1987) and Jeffreys (1961) use this concept and refer to it as 'orthogonality'. We prefer the term *information-orthogonality* to distinguish it from the other orthogonality concepts and to stress that it is defined in terms of the properties of the information matrix. See Tisbshirani and Wasserman (1994) and Woutersen (2000) for an overview of orthogonality concepts. Consider the log likelihood $L(\beta, f(\beta, \lambda))$ where the nuisance parameter f is written as a function of β and the orthogonal nuisance parameter λ. Differentiating $L(\beta, f(\beta, \lambda))$ with respect to β and λ yields

$$\begin{aligned} \frac{\partial L(\beta, f(\beta, \lambda))}{\partial \beta} &= L_\beta + L_f \frac{\partial f}{\partial \beta} \\ \frac{\partial^2 L(\beta, f(\beta, \lambda))}{\partial \lambda \partial \beta} &= L_{f\beta} \frac{\partial f}{\partial \lambda} + L_{ff} \frac{\partial f}{\partial \lambda} \frac{\partial f}{\partial \beta} + L_f \frac{\partial^2 f}{\partial \lambda \partial \beta} \end{aligned}$$

where L_f is a score and therefore $EL_f = 0$. Information orthogonality requires that the cross-derivative $\frac{\partial^2 L(\beta, f(\beta,\lambda))}{\partial\lambda\partial\beta}$ is zero in expectation, i.e.

$$EL_{\beta\lambda} = EL_{f\beta} \frac{\partial f}{\partial \lambda} + EL_{ff} \frac{\partial f}{\partial \lambda} \frac{\partial f}{\partial \beta} = 0.$$

[1] As $N \to \infty$, using the marginal posteriors is asymptotically equivalent. Considering the mode of the posterior, however, simplifies the algebra.

This condition implies the following differential equation

$$EL_{f\beta} + EL_{ff}\frac{\partial f}{\partial \beta} = 0. \tag{2}$$

If equation (2) has an analytical solution then $L(\beta, f(\beta,\lambda))$ is an explicit function of $\{\beta,\lambda\}$ and we refer to such a parametrization as an *explicit parametrization.* In most cases, however, equation (2) has an implicit solution and we have to recover the Jacobian $\frac{\partial\lambda}{\partial f}$ from this implicit solution. In this case, $L(\beta,\lambda)$ has an *implicit parametrization.* The information-orthogonal parametrization of the dynamic linear model without regressors is explicit and given by Lancaster (1997). We need an implicit parametrization to deal with regressors and will derive that parametrization in the next section. For both implicit and explicit parametrization, Woutersen (2001) shows that the integrated likelihood estimator is adaptive if $T \propto N^{\alpha}$ $\alpha > \frac{1}{3}$ and $\hat{\beta} - \beta$ is $O_p(T^{-2})$ if $\alpha \leq \frac{1}{3}$.

3 Orthogonality in the Dynamic Linear Model

Consider the dynamic linear model with fixed effects without regressors,

$$y_{is} = y_{i,s-1}\rho + f_i + \varepsilon_{is} \text{ where } E\varepsilon_{is} = 0, E\varepsilon_{is}^2 = \sigma^2,\ E\varepsilon_{is}\varepsilon_{it} = 0 \text{ for } s \neq t \text{ and } s = 1,...,T.$$

Lancaster (2000) conditions on y_{i0} and suggests the following parametrization[2]

$$f_i = y_{i0}(1-\rho) + \lambda_i e^{-b(\rho)} \text{ where } b(\rho) = \frac{1}{T}\sum_{s=1}^{T}\frac{T-s}{s}\rho^s.$$

Analogue to quasi-maximum likelihood estimators, normality of the error terms is assumed in order to derive the integrated likelihood estimator. The estimator depends only on the first two moments of y_{is} and is given by Lancaster (1997). In particular, the score of the integrated likelihood has the following form (see apendix 1 for derivation),

$$\begin{aligned} L_{\rho}^{i,I} &= b'(\rho) + \frac{1}{\sigma^2}\sum_s (y_s - y_{s-1}\rho)y_{s-1} - \frac{1}{\sigma^2}T(\overline{y_s - y_{s-1}\rho})\overline{y_{s-1}} \\ L_{\sigma^2}^{i,I} &= \frac{1}{\sigma^2}\{\frac{T-1}{2} - \frac{1}{2\sigma^2}\sum_s (y_s - y_{s-1}\rho)^2 + \frac{T}{2}(\overline{y_s - y_{s-1}\rho})^2\} \end{aligned}$$

where $b(\rho)' = \frac{1}{T}\sum_{s=1}^{T}(T-s)\rho^{s-1}$.

[2] Appendix 5 of Woutersen (2001) gives information-orthogonal parametrizations for linear models with more then one autoregressive term.

3.1 Dynamic linear model with regressors

Now consider the dynamic linear model with fixed effects and regressors,

$y_{is} = y_{i,s-1}\rho + x_{is}\beta + f_i + \varepsilon_{is}$ where $E\varepsilon_{is} = 0, E\varepsilon_{is}^2 = \sigma^2,\ E\varepsilon_{is}\varepsilon_{it} = 0$ for $s \neq t$ and $s = 1, ..., T$.

We assume normality in order to derive the following likelihood contribution for individual i.

$$L^i = -T\ln\sigma - \frac{1}{2\sigma^2}\sum_s (y_{is} - y_{i,s-1}\rho - x_{is}\beta - f_i)^2.$$

It follows from Woutersen (2001, appendix 5) that the information-orthogonal nuisance parameter has the following expression,

$$\lambda = \frac{1}{T}\sum_s E\int_{-\infty}^{y_{i,s-1}\rho + x_{is}\beta + f_i} d\mu.$$

Thus, λ is expressed as an unbounded integral. Fortunately, its derivatives are finite,

$$\begin{aligned} \frac{\partial\lambda}{\partial f} &= 1 \\ \frac{\partial\lambda}{\partial\rho} &= \frac{1}{T}\sum_s Ey_{i,s-1} \\ \frac{\partial\lambda}{\partial\sigma^2} &= 0. \end{aligned}$$

Normalizing the regressors to have mean zero, $\sum_s x_{is} = 0$, gives

$$\frac{\partial\lambda}{\partial\beta} = \sum_s x_{is} = 0$$

and

$$\frac{df}{d\rho} = -\frac{\frac{\partial\lambda}{\partial\rho}}{\frac{\partial\lambda}{\partial f}} = -\frac{1}{T}\sum_s Ey_{i,s-1}.$$

Thus,

$$L_\rho^{i,I}(\beta,\rho,\sigma^2) = \frac{d\ln\int e^{L^i}d\lambda}{d\rho} = \frac{\int(L_\rho^i + L_f^i\frac{df}{d\rho})e^{L^i}df}{\int e^{L^i}df} \tag{3}$$

$$L_\beta^{i,I}(\beta,\rho,\sigma^2) = \frac{d\ln\int e^{L^i}d\lambda}{d\beta} = \frac{\int L_\beta^i e^{L^i}df}{\int e^{L^i}df} \tag{4}$$

$$L_{\sigma^2}^{i,I}(\beta,\rho,\sigma^2) = \frac{1}{\sigma^2}\{\frac{T-1}{2} - \frac{1}{2\sigma^2}\sum_s (y_{is} - y_{i,s-1}\rho - x_{is}\beta)^2 + \frac{T}{2}(\overline{y_{is} - y_{i,s-1}\rho - x_{is}\beta})^2\}.$$

Thus, the estimate for the variance is

$$\widehat{\sigma^2} = \frac{1}{N(T-1)} \sum_i \{ \sum_s (y_{is} - y_{i,s-1}\rho - x_{is}\beta)^2 + T(\overline{y_{is} - y_{i,s-1}\rho - x_{is}\beta})^2 \}.$$

4 Adaptiveness

Woutersen (2001) shows that the integrated likelihood estimator is adaptive in the sense that it is equivalent to the infeasible maximum likelihood estimator that assumes the nuisance parameters to be known. The conditions for this result are a regularity condition that the integrated likelihood can be approximated by a Laplace formula and the substantial condition that $T \propto N^{\alpha}$ where $\alpha > \frac{1}{3}$. Suppose that T is rather small so that an asymptotics in which $T \propto N^{\alpha}$ where $\alpha > \frac{1}{3}$ is not satisfactory. Woutersen (2001) gives sufficient conditions for the integrated likelihood estimator to be consistent for T fixed and $N \to \infty$. We now show that consistency for fixed T implies that the integrated likelihood estimator is adaptive for an asymptotics with T increasing.

Proposition (case without regressors)

Suppose the asymptotic variance of $\rho_{ML} = \arg\max_{\rho} L(\rho, \lambda_0)$ *equals* $\Psi = \left[\frac{1}{NT} E\{L_{\rho} L_{\rho}'\}\right]^{-1}$ *and that* $T \propto N^{\alpha}$ *where* $\alpha > 0$. *Then the integrated likelihood estimator* $\widehat{\rho}$ *is an adaptive estimator and*

$$\sqrt{NT}(\widehat{\rho}_I - \rho_0) \to_d N(0, \Psi).$$

Proof: See appendix 2 .

5 Simulation Results

We use the same simulation designs as Hahn, Hausman and Kuersteiner (2001), (HHK), Hahn and Kuersteiner (2002), (HK), and Kiviet (1995). When we compare the performances of the integrated likelihood estimator with the performances of HHK estimator we considered the case without regressors, looking only at the performances of the autoregressive parameter estimator (ρ). We designed the Monte Carlo experiment by looking at values of the autoregressive parameter $\rho = \{0.1, 0.3, 0.5, 0.8, 0.9, 0.95, 0.99\}$ for samples of size $n = \{100, 500\}$ and $T = \{5, 10\}$ and assuming unknown σ^2. Thus, we extended the analysis of HHK by looking at values of ρ closed to the unit root. The number of simulation replications was set at 5000. The simulations results are presented in Table.1.

Looking at the $RMSE$ of the estimates, we observe that the integrated likelihood estimator is performing better than the other estimators in all situations. Also, we observe that for small values of $\rho = \{0.3, 0.5\}$, for $n = \{500\}$ and $T = \{10\}$, the bias for integrated likelihood estimator is slighty higher than the bias of HHK estimator , and that in any other cases the bias is lower. We can observe very good performances of the integrated likelihood estimator in the vicinity of unit root. Thus, in Table 3, we presented the performances of the integrated likelihood estimator for high values of $\rho = \{0.75, 0.8, 0.85, 0.9, 0.95\}$ for $n = 100$ and $T = 5$. The results show that our estimator is performing better in terms of mean and median bias, exception at $\rho = 0.9$ for the mean bias, and much better in terms of MSE in comparison with the other available estimators. Thus, our results confirm the prediction about the MSE in comparison to the other available estimators.

Also, we did Monte Carlo simulations using the design of HK (2002), where $\rho = \{0.0, 0.3, 0.6, 0.9\}$, $n = \{100, 200\}$ and $T = \{5, 10, 20\}$. Our simulations results are presented in Table 2. In all situations our results are presenting lower bias and lower $RMSE$. Thus, even for the cases where we have the same $RMSE$ ($\rho = \{0.0, 0.3\}$ for $n = 200$ and $T = 20$) , our results are better due to lower bias.

When we compare the performances of the integrated likelihood estimator with the performances of the Kieviet estimator we are considering the case with covariates. We also compare the results of the integrated likelihood estimator using Lancaster's (1997) approximation, to the Kieviet results, see Table 3.

We are considering 10 out of 14 cases Kieviet considered, this is due to the fact that we are not assuming any distribution for the fixed effect as Kieviet assumed. Thus, we present results for 10 different parameter combinations which are matching Kieviet's combinations for $\mu = 1$. The results for 5000 replications are presented in Table 4.

Comparing the results obtained using Lancaster's (1997) approximation to the Kieviet' results, we observe that for values of $\rho = 0$, Kiviet's estimator is performing better than integrated likelihood estimator and as ρ increases the reverse is true, exception is at $\rho = 0.4$ where the bias of the integrated likelihood ρ estimator is higher than Kiviet's at $\rho = 0.8$, where $RMSE$ of the integrated likelihood β estimator is larger than Kieviet's.

Looking at the integrated likelihood estimator results obtained by using the implicit parametrization of Woutersen (2001) , see Table 5, we observe improvements, thus we are not finding any cases for which Kiviet.s estimator is doing better than the integrated likelihood estimator.

Overall, the simulations show that the integrated likelihood estimator is very good in terms of MSE and bias, see section 9 for the tables.

6 Conclusion

This paper derived new adaptiveness result for the integrated likelihood estimator. The integrated likelihood estimator is adaptive for an asymptotic with T increasing. Simulations show the relevance of the adaptiveness results since the MSE of the integrated likelihood estimator was smaller then the MSE of competing estimators for several versions of the dynamic linear model with fixed effects. Also, our simulation results showed very good performances of the estimator for slower adjustment processes, ρ close to unit root.

Then, we derived a new estimator for the dynamic linear model with fixed effects and regressors. The new estimator performed very well, showing in simulations very good results for both MSE and bias.

7 Appendices

Appendix 1.

We assume normality of the error term $\left(\varepsilon_{is} \sim N\left(0, \sigma^2\right)\right)$ to derive moment coditions. Defining the integrated likelihood by $e^{L^{i,I}}$:

$$
\begin{aligned}
e^{L^{i,I}} &= \frac{1}{\sigma^T} e^{b(\rho)} \int e^{-\frac{1}{2\sigma^2}\sum_s (y_s - y_{s-1}\rho - f)^2} df \\
&= \frac{1}{\sigma^T} e^{b(\rho)} \int e^{-\frac{1}{2\sigma^2}\sum_s (y_s - y_{s-1}\rho - f)^2} df \\
&= \frac{1}{\sigma^T} e^{b(\rho) - \frac{1}{2\sigma^2}\sum_s (y_s - y_{s-1}\rho)^2} \int e^{-\frac{T}{2\sigma^2}\{f^2 - 2f(\overline{y_s - y_{s-1}\rho})\}} df \\
&\propto \frac{1}{\sigma^{T-1}} e^{b(\rho) - \frac{1}{2\sigma^2}\sum_s (y_s - y_{s-1}\rho)^2 + \frac{T}{2}(\overline{y_s - y_{s-1}\rho})^2}
\end{aligned}
$$

This implies that the log-of the integrated likelihood for individual i and the moments we are using to estimate parameters are:

$$
\begin{aligned}
L^{i,I} &= \frac{T-1}{2}\ln(\sigma^2) + b(\rho) - \frac{1}{2\sigma^2}\sum_s (y_s - y_{s-1}\rho)^2 + \frac{T}{2\sigma^2}(\overline{y_s - y_{s-1}\rho})^2 \\
L^{i,I}_{\rho} &= b'(\rho) + \frac{1}{\sigma^2}\{\sum_s (y_s - y_{s-1}\rho) y_{s-1} - T(\overline{y_s - y_{s-1}\rho})\overline{y_{s-1}}\} \\
L^{i,I}_{\rho\rho} &= b''(\rho) - \frac{1}{\sigma^2}\sum_s y_{s-1}^2 + \frac{T}{\sigma^2}\,\overline{y_{s-1}}^2 \\
L^{i,I}_{\sigma^2} &= \frac{1}{\sigma^2}\{\frac{T-1}{2} - \frac{1}{2\sigma^2}\sum_s (y_s - y_{s-1}\rho)^2 + \frac{T}{2}(\overline{y_s - y_{s-1}\rho})^2\}
\end{aligned}
$$

The functions $L^{i,I}_{\rho}$ and $L^{i,I}_{\sigma^2}$ can be used as a moment function.We can derive the same result when we integrate the score function:

$$
\begin{aligned}
L^{i,I}_{\rho} &= \frac{\int L^i_{\rho} e^L d\lambda}{\int e^L d\lambda} = \frac{\int L^i_{\rho} e^L df}{\int e^L df} \text{ since } \frac{\partial \lambda}{\partial f} = e^{b(\rho)} \text{ does not depend on } \lambda. \\
\text{constant } e^L &= \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2}\{\sum_s (y_s - y_{s-1}\rho)^2\} - \frac{T}{2}\{f^2 - 2f(\overline{y_s - y_{s-1}\rho})\}} \\
&= \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2}\{\sum_s (y_s - y_{s-1}\rho)^2\} + \frac{T}{2}(\overline{y_s - y_{s-1}\rho})^2 - \frac{T}{2}(f - \overline{y_s - y_{s-1}\rho})^2}
\end{aligned}
$$

$$\text{Note that } \frac{\int f e^L df}{\int e^L df} = \tilde{y}_s - \tilde{y}_{s-1}\rho \text{ and } \frac{\int (f - (\tilde{y}_s - \tilde{y}_{s-1}\rho))^2 e^L df}{\int e^L df} = \frac{1}{T}$$

$$L_{\rho}^{i,I} = b'(\rho) + \sum_s (y_s - y_{s-1}\rho) y_{s-1} - T(\overline{y_s - y_{s-1}\rho})\overline{y_{s-1}}$$

$$L_{\rho\rho}^{i,I} = b''(\rho) - \sum_s y_{s-1}^2 + T\, \overline{y_{s-1}}^2$$

Appendix 2. Proposition

To be shown:

$$\sqrt{NT}(\widehat{\rho}_I - \rho_0) \to_d N(0, \Psi),$$

where Ψ is the asymptotic variance of the ML estimator.

Proof:

We are considering the case without regressors.

a) Determining the asymptotic variance of the ML estimator for orthogonal fixed effect

We assume normality of the error term $\left(\varepsilon_{is} \sim N\left(0, \sigma^2\right)\right)$ to derive moment coditions. After deriving the moment conditions we show that under $E\varepsilon_{is} = 0$, $E\varepsilon_{is}^2 = \sigma^2$ and $E\varepsilon_{is}\varepsilon_{ij} = 0$ for $j \neq s$ our Proposition holds.

Define the known orthogonal fixed effects by λ_i, with initial fixed effects defined by $f_i = (1-\rho)\, y_{i0} - \lambda_i e^{-b(\rho)}$.

If we define the loglikelihood with known orthogonal fixed effects for individual i by L^i, we have:

$$L^i = -\frac{1}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_s (\tilde{y}_s - \tilde{y}_{s-1}\rho - \lambda e^{-b(\rho)})^2 \text{ where } \tilde{y}_s = y_s - y_0.$$

Now, we can derive the first and second moments necessary to determine the asymptotic variance for ML estimator:

$$
\begin{aligned}
L_{\rho}^{i} &= \frac{1}{\sigma^2}\sum_s(\tilde{y}_s - \tilde{y}_{s-1}\rho - \lambda e^{-b(\rho)})(\tilde{y}_{s-1} - \lambda b'(\rho)e^{-b(\rho)}) \\
&= \frac{1}{\sigma^2}\sum_s(\tilde{y}_s - \tilde{y}_{s-1}\rho - \lambda e^{-b(\rho)})(\tilde{y}_{s-1} - \lambda b'(\rho)e^{-b(\rho)}) \\
L_{\rho\rho}^{i} &= -\frac{1}{\sigma^2}\sum_s(\tilde{y}_{s-1} - \lambda b'(\rho)e^{-b(\rho)})^2 + \frac{\lambda}{\sigma^2}\sum_s(\tilde{y}_s - \tilde{y}_{s-1}\rho - \lambda e^{-b(\rho)})\left(b'^2(\rho) - b''(\rho)\right)e^{-b(\rho)}
\end{aligned}
$$

where $b(\rho) = \frac{1}{T}\sum_{s=1}^{T}\frac{T-s}{s}\rho^s$, $b'(\rho) = \frac{1}{T}\sum_{s=1}^{T}(T-s)\rho^{s-1}$ and $b''(\rho) = \frac{1}{T}\sum_{s=1}^{T}(T-s)(s-1)\rho^{s-2}$.

Defining, by $\frac{\partial f}{\partial \rho} = -y_0 + \lambda b'(\rho)e^{-b(\rho)}$ we can rewrite:

$$
\begin{aligned}
L_{\rho\rho}^{i} &= -\frac{1}{\sigma^2}\sum_s(y_{s-1} + \frac{\partial f}{\partial \rho})^2 + \frac{\lambda}{\sigma^2}\sum_s \varepsilon_s\left(b'^2(\rho) - b''(\rho)\right)e^{-b(\rho)} \\
&= -\frac{1}{\sigma^2}\left(\sum_s y_{s-1}^2 + 2T\frac{\partial f}{\partial \rho}\overline{y}_{s-1} + T\left(\frac{\partial f}{\partial \rho}\right)^2\right) + \frac{\lambda}{\sigma^2}\sum_s \varepsilon_s\left(b'^2(\rho) - b''(\rho)\right)e^{-b(\rho)} \\
L_{\rho\rho}^{i} &= -\frac{1}{\sigma^2}\sum_s y_{s-1}^2 - \frac{2T}{\sigma^2}\frac{\partial f}{\partial \rho}\overline{y}_{s-1} - \frac{T}{\sigma^2}\left(\frac{\partial f}{\partial \rho}\right)^2 + \frac{\lambda}{\sigma^2}\sum_s \varepsilon_s\left(b'^2(\rho) - b''(\rho)\right)e^{-b(\rho)}.
\end{aligned}
$$

Taking expectations we have:

$EL_{\rho\rho}^{i} = -\frac{1}{\sigma^2}E\sum_s y_{s-1}^2 - \frac{2T}{\sigma^2}\frac{\partial f}{\partial \rho}E\overline{y}_{s-1} - \frac{T}{\sigma^2}\left(\frac{\partial f}{\partial \rho}\right)^2 + \frac{\lambda}{\sigma^2}\sum_s E\left(\varepsilon_s\right)\left(b'^2(\rho) - b''(\rho)\right)e^{-b(\rho)}$

$EL_{\rho\rho}^{i} = -\frac{1}{\sigma^2}E\sum_s y_{s-1}^2 - \frac{2T}{\sigma^2}\frac{\partial f}{\partial \rho}E\overline{y}_{s-1} - \frac{T}{\sigma^2}\left(\frac{\partial f}{\partial \rho}\right)^2.$

Given that $E\varepsilon_{is} = 0$, $E\varepsilon_{is}^2 = \sigma^2$ and $E\varepsilon_{is}\varepsilon_{ij} = 0$ for $j \neq s$ and

$$
\begin{aligned}
L_{\rho}^{i} &= \frac{1}{\sigma^2}\sum_s(y_{is} - y_{i,s-1}\rho - f_i)(y_{i,s-1} + \frac{\partial f_i}{\partial \rho}) \\
&= \frac{1}{\sigma^2}\sum_s \varepsilon_{is}(y_{i,s-1} + \frac{\partial f_i}{\partial \rho}) \text{ and} \\
\left(L_{\rho}^{i}\right)\left(L_{\rho}^{i}\right)' &= \frac{1}{\sigma^4}\left(\sum_s \varepsilon_{is}(y_{i,s-1} + \frac{\partial f_i}{\partial \rho})\right)^2 \\
&= \frac{1}{\sigma^4}\sum_s \varepsilon_{is}^2(y_{i,s-1} + \frac{\partial f_i}{\partial \rho})^2 + \frac{2}{\sigma^4}\sum_{j\neq s}\left(\varepsilon_{is}\right)\left(\varepsilon_{ij}\right)\left(y_{i,s-1} + \frac{\partial f_i}{\partial \rho}\right)\left(y_{i,j-1} + \frac{\partial f_i}{\partial \rho}\right),
\end{aligned}
$$

we have $E\left(\left(L_{\rho}\right)\left(L_{\rho}\right)'\right)$ that is

$$\begin{aligned} E\left(\left(L_{\rho}\right)\left(L_{\rho}\right)'\right) &= \frac{1}{\sigma^4}\sum_i\sum_s E\left(\varepsilon_{is}^2\left(y_{i,s-1}+\frac{\partial f_i}{\partial\rho}\right)^2\right) \\ &\quad +\frac{2}{\sigma^4}\sum_{j\neq s}E\left(\left(\varepsilon_{is}\right)\left(\varepsilon_{ij}\right)\left(y_{i,s-1}+\frac{\partial f_i}{\partial\rho}\right)\left(y_{i,j-1}+\frac{\partial f_i}{\partial\rho}\right)\right) \\ &= \frac{1}{\sigma^4}\sum_i\sum_s E\left(\varepsilon_{is}^2\left(y_{i,s-1}+\frac{\partial f_i}{\partial\rho}\right)^2\right)=\frac{\sigma^2}{\sigma^4}\sum_i\sum_s E\left(y_{i,s-1}+\frac{\partial f_i}{\partial\rho}\right)^2 \\ &= \frac{1}{\sigma^2}\sum_i\sum_s Ey_{i,s-1}^2+\frac{2T}{\sigma^2}\frac{\partial f_i}{\partial\rho}\sum_i E\overline{y}_{i-}+\frac{T}{\sigma^2}\sum_i\left(\frac{\partial f_i}{\partial\rho}\right)^2. \end{aligned}$$

Thus,

$$E\left(\left(L_{\rho}\right)\left(L_{\rho}\right)'\right)=-EL_{\rho\rho},$$

therefore, the asymptotic variance of ML estimator for orthogonal fixed effects will be defined as:

$$\begin{aligned} \Psi &= Asy.var\left(\widehat{\rho}_{ML}\right)=\left[\frac{1}{NT}EL_{\rho\rho}\right]^{-1}\left[\frac{1}{NT}E\left(\left(L_{\rho}\right)\left(L_{\rho}\right)'\right)\right]\left[\frac{1}{NT}EL_{\rho\rho}\right]^{-1} \\ &= \left[\frac{1}{NT}E\left(\left(L_{\rho}\right)\left(L_{\rho}\right)'\right)\right]^{-1}=\left[-\frac{1}{NT}EL_{\rho\rho}\right]^{-1} \\ \Psi &= Asy.var\left(\widehat{\rho}_{ML}\right)=\left[-\frac{1}{NT}EL_{\rho\rho}\right]^{-1} \end{aligned}$$

where $EL_{\rho\rho}$ is

$$EL_{\rho\rho}=-\frac{1}{\sigma^2}\sum_i\sum_s Ey_{i,s-1}^2-\frac{2T}{\sigma^2}\frac{\partial f_i}{\partial\rho}\sum_i E\overline{y}_{i-}-\frac{T}{\sigma^2}\sum_i\left(\frac{\partial f_i}{\partial\rho}\right)^2.$$

b) Determining the asymptotic variance of the integrated likelihood estimator

Assuming again $\varepsilon_{is}\sim N\left(0,\sigma^2\right)$ we derived the moment conditions for the integrated likelihood estimator. If we define the integrated likelihood for individual i by $e^{L^{i,I}}$, we have

$$e^{L^{i,I}}\propto\frac{1}{\sigma^{T-1}}e^{b(\rho)-\frac{1}{2\sigma^2}\sum_s(y_s-y_{s-1}\rho)^2+\frac{T}{2\sigma^2}(\overline{y_s-y_{s-1}\rho})^2}.$$

This implies that the log of integrated likelihood to be:

$$L^{i,I}=\frac{T-1}{2}\ln(\sigma^2)+b(\rho)-\frac{1}{2\sigma^2}\sum_s(y_s-y_{s-1}\rho)^2+\frac{T}{2\sigma^2}(\overline{y_s-y_{s-1}\rho})^2$$

and the corresponding first and second moments for the integrated likelihood are:

$$\begin{aligned} L_{\rho}^{i,I} &= b'(\rho) + \frac{1}{\sigma^2}\{\sum_s (y_s - y_{s-1}\rho)y_{s-1} - T(\overline{y_s - y_{s-1}\rho})\overline{y_{s-1}}\} \\ L_{\rho\rho}^{i,I} &= b''(\rho) - \frac{1}{\sigma^2}\sum_s y_{s-1}^2 + \frac{T}{\sigma^2}\,\overline{y_{s-1}}^2. \end{aligned}$$

Taking expectations we have:

$$EL_{\rho\rho}^{i,I} = b''(\rho) - \frac{1}{\sigma^2}\sum_s Ey_{s-1}^2 + \frac{T}{\sigma^2}\,E\overline{y_{s-1}}^2.$$

Relying on the work of Newey, W.K. and D. McFadden (1994) on the asymptotic variance of moment estimators, we can express the asymptotic variance of the integrated likelihood estimator as:

$$Asy.var\,(\widehat{\rho}_I) = \left[\frac{1}{NT}EL_{\rho\rho}^I\right]^{-1}\left[\frac{1}{NT}E\left(\left(L_{\rho}^I\right)\left(L_{\rho}^I\right)'\right)\right]\left[\frac{1}{NT}EL_{\rho\rho}^I\right]^{-1}$$

where $EL_{\rho\rho}^I$ is

$$EL_{\rho\rho}^I = Nb''(\rho) - \frac{1}{\sigma^2}E\left(\sum_i\sum_s y_{i,s-1}^2\right) + \frac{T}{\sigma^2}\,E\left(\sum_i \overline{y_{i-}}^2\right)$$

and

$$EL_{\rho}^I = Nb'(\rho) + \frac{1}{\sigma^2}E\left(\sum_i\{\sum_s (y_{is} - y_{i,s-1}\rho)y_{i,s-1} - T(\overline{y_{is} - y_{i,s-1}\rho})\overline{y_{i,s-1}}\}\right).$$

Given the fact that $\frac{1}{NT}E\left(\left(L_{\rho}^I\right)\left(L_{\rho}^I\right)'\right) = \frac{1}{NT}E\sum_i\left(L_{\rho}^{i,I}\right)\left(L_{\rho}^{i,I}\right)'$ when $EL_{\rho}^I = 0$, we will compute first:

$$
\begin{aligned}
\left(L_{\rho}^{i,I}\right)\left(L_{\rho}^{i,I}\right)' &= b'(\rho)^2 + \frac{2b'(\rho)}{\sigma^2}\sum_s (y_s - y_{s-1}\rho)y_{s-1} - \frac{2b'(\rho)}{\sigma^2}T\overline{(y_s - y_{s-1}\rho)}\,\overline{y_{s-1}} \\
&\quad + \frac{1}{\sigma^4}\left(\sum_s (y_s - y_{s-1}\rho)y_{s-1} - T\overline{(y_s - y_{s-1}\rho)}\,\overline{y_{s-1}}\right)^2 \\
&= b'(\rho)^2 + \frac{2b'(\rho)}{\sigma^2}\sum_s (f+\varepsilon_s)\,y_{s-1} - \frac{2b'(\rho)}{\sigma^2}T\left(f+\overline{\varepsilon_s}\right)\overline{y_{s-1}} \\
&\quad + \frac{1}{\sigma^4}\left(\sum_s (f+\varepsilon_s)\,y_{s-1} - T\left(f+\overline{\varepsilon_s}\right)\overline{y_{s-1}}\right)^2 \\
&= b'(\rho)^2 + \frac{2b'(\rho)}{\sigma^2}\left(\sum_s (\varepsilon_s)(y_{s-1}) - T\left(\overline{\varepsilon_s}\right)\left(\overline{y_{s-1}}\right)\right) \\
&\quad + \frac{1}{\sigma^4}\left(\sum_s (\varepsilon_s)\,y_{s-1} - T\left(\overline{\varepsilon_s}\right)\overline{y_{s-1}}\right)^2 .
\end{aligned}
$$

Thus, $E\left(L_{\rho}^{I}\right)\left(L_{\rho}^{I}\right)'$ is determined by:

$$
\begin{aligned}
E\left(L_{\rho}^{I}\right)\left(L_{\rho}^{I}\right)' &= Nb'(\rho)^2 + \frac{2b'(\rho)}{\sigma^2}E\sum_i\left(\sum_s (\varepsilon_{is})(y_{i,s-1}) - T\left(\overline{\varepsilon_{is}}\right)\left(\overline{y_{i,s-1}}\right)\right) \\
&\quad + \frac{1}{\sigma^4}E\sum_i\left(\sum_s (\varepsilon_{is})\,y_{i,s-1} - T\left(\overline{\varepsilon_{is}}\right)\overline{y_{i,s-1}}\right)^2 ,
\end{aligned}
$$

where $b'(\rho) = \frac{1}{T}\sum_{s=1}^{T}(T-s)\rho^{s-1}$; $b'(\rho)^2 = \left(\frac{1}{T}\sum_{s=1}^{T}(T-s)\rho^{s-1}\right)^2$,

$$
\begin{aligned}
\lim_{T\to\infty} b'(\rho)^2 &= \lim_{T\to\infty}\left(\frac{1}{T}\sum_{s=1}^{T}(T-s)\rho^{s-1}\right)^2 \\
&= \lim_{T\to\infty}\left(\sum_{s=1}^{T}\rho^{s-1} - \frac{1}{T}\sum_{s=1}^{T}s\rho^{s-1}\right)^2 \\
&= \lim_{T\to\infty}\left(\sum_{s=1}^{T}\rho^{s-1}\right)^2 - \lim_{T\to\infty}\frac{2}{T}\sum_{s=1}^{T}\rho^{s-1}\sum_{s=1}^{T}s\rho^{s-1} + \lim_{T\to\infty}\left(\frac{1}{T}\sum_{s=1}^{T}s\rho^{s-1}\right)^2 \\
\lim_{T\to\infty} b'(\rho)^2 &= \frac{1}{(1-\rho)^2} + O\left(T^{-1}\right) + O(T^{-2}).
\end{aligned}
$$

and

$$\begin{aligned}
\lim_{T\to\infty} b''(\rho) &= \lim_{T\to\infty}\left(\frac{1}{T}\sum_{s=1}^{T}(T-s)(s-1)\rho^{s-2}\right) \\
&= \lim_{T\to\infty}\left(\frac{1}{T}\sum_{s=1}^{T}\left(Ts-T-s^2+s\right)\rho^{s-2}\right) \\
&= \lim_{T\to\infty}\sum_{s=1}^{T}s\rho^{s-2}-\lim_{T\to\infty}\sum_{s=1}^{T}\rho^{s-2}-\lim_{T\to\infty}\frac{1}{T}\sum_{s=1}^{T}s^2\rho^{s-2}+\lim_{T\to\infty}\frac{1}{T}\sum_{s=1}^{T}s\rho^{s-2} \\
&= \frac{1}{\rho}+\lim_{T\to\infty}\sum_{s=2}^{T}(s)\rho^{s-2}-\frac{1}{\rho}-\lim_{T\to\infty}\sum_{s=2}^{T}\rho^{s-2}+O\left(T^{-1}\right) \\
&= \lim_{T\to\infty}\sum_{s=2}^{T}(s-1+1)\rho^{s-2}-\frac{1}{1-\rho}+O\left(T^{-1}\right) \\
&= \lim_{T\to\infty}\sum_{s=2}^{T}(s-1)\rho^{s-2}+\lim_{T\to\infty}\sum_{s=2}^{T}\rho^{s-2}-\frac{1}{1-\rho}+O\left(T^{-1}\right) \\
&= \frac{1}{(1-\rho)^2}+\frac{1}{1-\rho}-\frac{1}{1-\rho}+O\left(T^{-1}\right)=\frac{1}{(1-\rho)^2}+O\left(T^{-1}\right).
\end{aligned}$$

The second moment of the integrated likelihood estimator is defined as:

$$L^I_{\rho\rho}=Nb''(\rho)-\frac{1}{\sigma^2}\sum_i\sum_s y^2_{i,s-1}+\frac{T}{\sigma^2}\sum_i \overline{y_{i,s-1}}^2$$

and after taking expectations we have

$$EL^I_{\rho\rho}=Nb''(\rho)-\frac{1}{\sigma^2}E\sum_i\sum_s y^2_{i,s-1}+\frac{T}{\sigma^2}E\sum_i \overline{y_{i,s-1}}^2.$$

Adding up the two moments we have:

$$\begin{aligned}
\tfrac{1}{NT}E\left(L^I_\rho\right)\left(L^I_\rho\right)'+\tfrac{1}{NT}EL^I_{\rho\rho} &= \tfrac{b'(\rho)^2}{T}+\tfrac{2b'(\rho)}{NT\sigma^2}E\textstyle\sum_i\left(\sum_s(\varepsilon_{is})(y_{i,s-1})-T\left(\overline{\varepsilon_{is}}\right)\left(\overline{y_{i,s-1}}\right)\right) \\
&+\tfrac{1}{NT\sigma^4}E\textstyle\sum_i\left(\sum_s(\varepsilon_{is})\,y_{i,s-1}-T\left(\overline{\varepsilon_{is}}\right)\overline{y_{i,s-1}}\right)^2+\tfrac{b''(\rho)}{T} \\
&-\tfrac{1}{NT\sigma^2}\textstyle\sum_i\sum_s Ey^2_{i,s-1}+\tfrac{1}{N\sigma^2}\sum_i E\,\overline{y_{s-1}}^2 \\
&=\tfrac{b'(\rho)^2}{T}+\tfrac{b''(\rho)}{T}+\tfrac{2b'(\rho)}{NT\sigma^2}\textstyle\sum_i\left(\sum_s E(\varepsilon_{is})\left(y_{i,s-1}-\overline{y_{i,s-1}}\right)\right) \\
&+\tfrac{1}{NT\sigma^4}E\textstyle\sum_i\left(\sum_s(\varepsilon_{is})\left(y_{i,s-1}-\overline{y_{i,s-1}}\right)\right)^2-\tfrac{1}{NT\sigma^2}\sum_i\sum_s Ey^2_{i,s-1} \\
&+\tfrac{1}{N\sigma^2}\textstyle\sum_i E\,\overline{y_{s-1}}^2 \\
&=O\left(T^{-1}\right)+O\left(T^{-1}\right)+\tfrac{1}{NT\sigma^4}\textstyle\sum_i\sum_s E(\varepsilon_{is})^2\left(y_{i,s-1}-\overline{y_{i,s-1}}\right)^2 \\
&+\tfrac{2}{NT\sigma^4}\textstyle\sum_i\sum_s\sum_{j\neq s}E(\varepsilon_{is})(\varepsilon_{ij})\left(y_{i,s-1}-\overline{y_{i,s-1}}\right)\left(y_{i,j-1}-\overline{y_{i,j-1}}\right)
\end{aligned}$$

$$-\frac{1}{NT\sigma^2}\sum_i\sum_s Ey_{i,s-1}^2+\frac{1}{N\sigma^2}\sum_i E\,\overline{y_{i,s-1}}^2$$

$$=O\left(T^{-1}\right)+O\left(T^{-1}\right)+\frac{\sigma^2}{NT\sigma^4}\sum_i\sum_s E\left(y_{i,s-1}-\overline{y_{i,s-1}}\right)^2$$

$$-\frac{1}{NT\sigma^2}\sum_i\sum_s Ey_{i,s-1}^2+\frac{1}{N\sigma^2}\sum_i E\,\overline{y_{i,s-1}}^2$$

$$\frac{1}{NT}E\left(L_\rho^I\right)\left(L_\rho^I\right)'+\frac{1}{NT}EL_{\rho\rho}^I=O\left(T^{-1}\right)+O\left(T^{-1}\right)+\frac{1}{NT\sigma^2}\sum_i\sum_s Ey_{i,s-1}^2$$

$$-\frac{1}{N\sigma^2}\sum_i E\,\overline{y_{i,s-1}}^2-\frac{1}{NT\sigma^2}\sum_i\sum_s Ey_{i,s-1}^2+\frac{1}{N\sigma^2}\sum_i E\,\overline{y_{i,s-1}}^2.$$

Thus, we get $\frac{1}{NT}E\left(L_\rho^I\right)\left(L_\rho^I\right)'+\frac{1}{NT}EL_{\rho\rho}^I$ that is $O\left(T^{-1}\right)$. This means that, when $T\to\infty$, we can write:

$$\frac{1}{NT}EL_{\rho\rho}^I=-\frac{1}{NT}E\left(\left(L_\rho^I\right)\left(L_\rho^I\right)'\right),$$

therefore, the asymptotic variance of the integrated likelihood estimator is given by

$$Asy.var\left(\widehat{\rho}_I\right)=\left[\frac{1}{NT}E\left(\left(L_\rho^I\right)\left(L_\rho^I\right)'\right)\right]^{-1}=\left[-\frac{1}{NT}EL_{\rho\rho}^I\right]^{-1}.$$

Comparing the two expectations $\frac{1}{NT}EL_{\rho\rho}^I$ and $\frac{1}{NT}EL_{\rho\rho}$ we can establish the rate at which the asymptotic variances are equal.

Taking the difference between $EL_{\rho\rho}^{i,I}-EL_{\rho\rho}^i$ we have:

$$\frac{1}{NT}EL_{\rho\rho}^I-\frac{1}{NT}EL_{\rho\rho}=\frac{b''(\rho)}{T}+\frac{1}{N\sigma^2}\sum_i E\overline{y_{i,s-1}}^2+\frac{2}{N\sigma^2}\sum_i\frac{\partial f_i}{\partial\rho}E\overline{y}_{i,s-1}+\frac{1}{N\sigma^2}\sum_i\left(\frac{\partial f_i}{\partial\rho}\right)^2$$

$$=\frac{b''(\rho)}{T}+\frac{1}{N\sigma^2}\sum_i\left(E\overline{y}_{i,s-1}+\frac{\partial f_i}{\partial\rho}\right)^2+\frac{1}{N\sigma^2}\sum_i Var\left(\overline{y}_{i,s-1}\right)$$

$$=\frac{b''(\rho)}{T}+\frac{1}{N\sigma^2}\sum_i\left(E\frac{\sum_s y_{i,s-1}}{T}+\frac{\partial f_i}{\partial\rho}\right)^2+\frac{1}{N\sigma^2}\sum_i Var\left(\frac{\sum_s y_{i,s-1}}{T}\right)$$

$$=\frac{b''(\rho)}{T}+\frac{1}{N\sigma^2}\sum_i\left(E\frac{\sum_s\left(\rho^{s-1}y_{i0}+(T-s+1)\rho^{s-1}f_i+\varepsilon_{i,s-1}\sum_{j=1}^{T-1}\rho^{j-s}\right)}{T}+\frac{\partial f_i}{\partial\rho}\right)^2$$

$$+\frac{1}{N\sigma^2}\sum_i Var\left(\frac{\sum_s\left(\rho^{s-1}y_{i0}+(T-s+1)\rho^{s-1}f_i+\varepsilon_{i,s-1}\sum_{j=1}^{T-1}\rho^{j-s}\right)}{T}\right)$$

Now, considering T is increasing, when $T\to\infty$, we have

$$\lim_{T\to\infty}E\overline{y}_{s-1}=\lim_{T\to\infty}E\frac{\sum_{s=1}^T\left(\rho^{s-1}y_{i0}+(T-s+1)\rho^{s-1}f_i+\varepsilon_{i,s-1}\sum_{j=1}^{T-1}\rho^{j-s}\right)}{T}$$

$$=\lim_{T\to\infty}\frac{\sum_s^T\rho^{s-1}y_{i0}}{T}+\lim_{T\to\infty}\frac{\sum_{s=1}^T(T-s+1)\rho^{s-1}f_i}{T}+\lim_{T\to\infty}\frac{\sum_{s=1}^T E\varepsilon_{i,s-1}\sum_{j=1}^{T-1}\rho^{j-s}}{T}$$

$$=\lim_{T\to\infty}\frac{\frac{1}{1-\rho}y_{i0}}{T}+\frac{1}{1-\rho}f_i-\lim_{T\to\infty}\frac{\sum_{s=1}^T(s-1)\rho^{s-1}f_i}{T}$$

$=\frac{1}{1-\rho}f_i+O\left(T^{-1}\right)$, because $\frac{\frac{1}{1-\rho}y_{i0}}{T}$ is $O\left(T^{-1}\right)$ and $\frac{\sum_{s=1}^T(s-1)\rho^{s-1}f_i}{T}$ is $O\left(T^{-1}\right)$.

Thus, $E\overline{y}_{s-1} \to \frac{f}{1-\rho}$ and $\frac{\partial f}{\partial \rho} = -fb'(\rho)$, but

$$\begin{aligned}
\lim_{T\to\infty} b'(\rho) &= \lim_{T\to\infty}\left(\frac{1}{T}\sum_{s=1}^{T}(T-s)\rho^{s-1}\right) \\
&= \lim_{T\to\infty}\sum_{s=1}^{T}\rho^{s-1} - \lim_{T\to\infty}\frac{1}{T}\sum_{s=1}^{T}s\rho^{s-1} \\
&= \frac{1}{1-\rho} - \lim_{T\to\infty}\frac{1}{T}\frac{1}{(1-\rho)^2} \\
\lim_{T\to\infty} b'(\rho) &= \frac{1}{1-\rho} + O\left(T^{-1}\right) \Longrightarrow \frac{\partial f}{\partial \rho} \to -\frac{f}{1-\rho} \text{ as } T\to\infty
\end{aligned}$$

Then, $\left(E\overline{y}_{s-1} + \frac{\partial f}{\partial \rho}\right)$ is $O\left(T^{-1}\right)$ and $\left(\overline{y}_{s-1} + \frac{\partial f}{\partial \rho}\right) \to 0$ as $T \to \infty$ and $\Rightarrow \left(E\overline{y}_{s-1} + \frac{\partial f}{\partial \rho}\right)^2$ *is* $O\left(T^{-2}\right)$. Also, if we are looking at $Var\left(\overline{y}_{i,s-1}\right)$ when $T \to \infty$ we have:

$$\begin{aligned}
Var\left(\overline{y}_{i,s-1}\right) &= Var\left(\frac{\sum_s\left(\rho^{s-1}y_{i0} + (T-s+1)\,\rho^{s-1}f_i + \varepsilon_{i,s-1}\sum_{j=1}^{T-1}\rho^{j-s}\right)}{T}\right) \\
&= \frac{1}{T^2}Var\left(\sum_{s=1}^{T}\left(\varepsilon_{i,s-1}\sum_{j=1}^{T-1}\rho^{j-s}\right)\right) = \frac{1}{T^2}\sum_{s=1}^{T}\left(\sigma^2\left(\sum_{j=1}^{T-1}\rho^{j-s}\right)^2\right)
\end{aligned}$$

and $\frac{1}{N\sigma^2}\sum_i Var\left(\overline{y}_{i,s-1}\right) = \frac{1}{NT}\sum_i\left(\frac{1}{T}\sum_{s=1}^{T}\left(\sum_{j=1}^{T-1}\rho^{j-s}\right)^2\right)$ is $O\left(T^{-1}\right)$.

Using the results derived, we have $\frac{1}{NT}EL^I_{\rho\rho} - \frac{1}{NT}EL_{\rho\rho}$ is $O\left(T^{-1}\right)$.

Given the fact that the difference between the asymptotic variances:

$$\begin{aligned}
Asy.var\left(\widehat{\rho}_I\right) - Asy.var\left(\widehat{\rho}_{ML}\right) &= \frac{1}{\frac{1}{NT}EL_{\rho\rho}} - \frac{1}{\frac{1}{NT}EL^I_{\rho\rho}} \\
Asy.var\left(\widehat{\rho}_I\right) - Asy.var\left(\widehat{\rho}_{ML}\right) &= \frac{\frac{1}{NT}EL^I_{\rho\rho} - \frac{1}{NT}EL_{\rho\rho}}{\left(\frac{1}{NT}EL_{\rho\rho}\right)\left(\frac{1}{NT}EL^I_{\rho\rho}\right)}
\end{aligned}$$

is of order $O\left(T^{-1}\right)$ or $o(1)$ as $N \to \infty$ and $T \propto N^{\alpha}$, where the only condition in α is $\alpha > 0$, we can conclude that the two asymptotic variances are equal:

$$Asy.var\left(\widehat{\rho}_I\right) = Asy.var\left(\widehat{\rho}_{ML}\right) = \Psi.$$

8 References

Alvarez, J. and M. Arellano (1998): "The Time Series and Cross-Section Asymptotics of Dynamic Panel Data Estimators," unpublished manuscript.

Arellano, M., and B. E. Honoré, (2001): "Panel Data Models: Some Recent Developments," in *Handbook of Econometrics,* Vol. 5, ed. by J. Heckman and E. Leamer, Amsterdam: North-Holland.

Baltagi, B. H. (1995): "*Econometric Analysis of Panel Data,*" John Wiley and Sons, New York.

Basu, D. (1977): "On the elimination of nuisance parameters," *Journal of the American Statistical Association,* 72, 355-366.

Berger, J. O., B. Liseo, and R. L. Wolpert (1999): "Integrated Likelihood Methods for Eliminating Nuisance Parameters," *Statistical Science*, 14, 1-28.

Bickel, P. J. (1982), "On Adaptive estimation", *Annals of Statistics,* vol.10, 647 - 671.

Bickel P. J., C. A. J. Klaassen, Y. Ritov and J. A. Wellner (1993), *Efficient and adaptive estimation for semiparametric models.* Johns Hopkins series in mathematical science.

Chamberlain, G. (1980): "Analysis of Covariance with Qualitative Data" *Review of Economic Studies*, 47, 225-238.

——— (1982): "Multivariate Regression for Panel Data," *Journal of Econometrics*, 18, 5-46.

——— (1984): "Panel Data," in *Handbook of Econometrics,* Vol. 2, ed. by Z. Griliches and M. D. Intriligator. Amsterdam: North-Holland.

——— (1985): "Heterogeneity, Omitted Variable Bias, and Duration Dependence," in *Longitudinal Analysis of Labor Market Data*, ed. by J. J. Heckman and B. Singer. Cambridge: Cambridge University Press.

Cox, D. R., and N. Reid (1987): "Parameter Orthogonality and Approximate Conditional Inference (with Discussion)," *Journal of the Royal Statistical Society*, Series B, 49, 1-39.

——— (1993): "A Note on the Calculation of Adjusted Profile Likelihood," *Journal of the Royal Statistical Society*, Series B, 45, 467-471.

Critchley, F. (1987): "Discussion of Parameter Orthogonality and Approximate Conditional Inference (by Cox, D. R., and N. Reid) *Journal of the Royal Statistical Society*, Series B, 49, 25-26.

Ferguson, H. (1991): "Asymptotic properties of conditional maximum-likelihood estimator," *The Canadian Journal of Statistics*, 20, 63-75.

Ferguson, H., N. Reid and D.R. Cox (1989): "Estimating equations from modified profile likelihood," in *Estimating Functions*, V.P. Godambe, ed., Oxford University Press.

Gelman, A., J. B. Carlin, H. S. Stern, D. B. Rubin (1995): *Bayesian Data Analysis,* Chapman and Hall.

Hahn, J., J. Hausman, and G. Kuersteiner (2001): "Bias Corrected Instrumental Variables Estimation for Dynamic Panel Models with Fixed Effects", MIT working paper.

Hahn, J. and G. Kuersteiner (2002): "Asymptotically Unbiased Inference for a Dynamic Panel

Model with Fixed Effects When Both n and T are Large", Forthcoming Econometrica.

Heckman, J. J. (1981a): "Statistical Models for Discrete Panel Data," in *Structural Analysis of Discrete Data with Econometric Applications*, edited by C. F. Manski and D. McFadden, pp. 114-178, MIT Press, Cambridge.

Heckman, J. J. (1981b): "The Incidental Parameter Problem and the Problem of Initial Conditions in Estimating a Discrete Time-Discrete Dtata Stochastic Process," in *Structural Analysis of Discrete Data with Econometric Applications*, edited by C. F. Manski and D. McFadden,

pp. 179-195, MIT Press, Cambridge.

Heckman, J., H. Ichimura, J. Smith and P. Todd (1998): "Characterizing Selection Bias Using Experimental Data," Econometrica, 66, 1017-1098.

Hsiao, C. (1986): *"Analysis of Panel Data".* Cambridge University Press, New York

Hsiao, C., M. H. Pesaran and A. K. Tahmiscioglu (1998): "Maximum Likelihood Estimation of Fixed Effects Dynamic Panel Data Models Covering Short Time Periods", USC working paper.

Hills, S. E. (1987): "Discussion of Parameter Orthogonality and Approximate Conditional Inference (by Cox, D. R., and N. Reid) *Journal of the Royal Statistical Society*, Series B, 49, 23-24.

Honoré, B. E. and E. Kyriazidou (2000): "Panel Data Discrete Choice Models with Lagged Dependent Variables," *Econometrica*, 68, 839-874.

Jeffreys, H. (1961): *Theory of Probability*, 3rd ed. Oxford: Clarendon Press.

Kass, R. E., L. Tierney, and J. B. Kadane (1990): "The Validity of Posterior Expansions Based on Laplace's Method," in *Essays in Honor of George Barnard*, eds. S. Geiser, S. J. Press, and A. Zellner, Amsterdam: North-Holland.

Kiviet (1995): "On Bias, Inconsistency and Efficiency of Various Estimators in Dynamic Panel Data Models", *Journal of Econometrics*, 1995, 68, 53-78.

Lancaster, T. (1997): "Orthogonal Parameters and Panel Data," Working paper, Brown University.

Lancaster, T. (1999): "Econometric Analysis of Dynamic Models: A Growth Theory Example,"

Working paper, Brown

University.

——— (1999): "Some Econometrics of Scarring," Brown manuscript, Brown University.

——— (2000): "The Incidental Parameters since 1948," *Journal of Econometrics,* 95, 391-413.

Liang, K. (1987): "Estimating functions and approximate conditional likelihood", *Biometrika*,

74, 695-702.

Mankiw, N.G., Romer D., and D. Weil: "A Contribution to the Empirics of Economic Growth,"

Quarterly Journal of Economics, CVII, 1992, 407-437.

Mundlak, Y. (1961): "Empirical Production Function Free of Management Bias," *Journal of Farm Economics,* 43, 44-56.

Nerlove, M. (2000): "The Future of Panel Data Econometrics," Working Paper University of Maryland.

Neyman, J., and E. L. Scott (1948): "Consistent Estimation from Partially Consistent Observations," *Econometrica*, 16, 1-32.

1. Newey, W. K and R. J. Smith (2001): "Higher Order Properties of GMM and Generalized

Empirical Likelihood Estimators," Working Paper MIT.

Newey, W. K (1990): "Semiparametric Efficiency Bounds," *Journal of Applied Econometrics*, 5, 99-135.

Newey, W. K., and D. McFadden (1994): "Large Sample Estimation and Hypothesis Testing," in *Handbook of Econometrics,* Vol. 4, ed. by R. F. Engle and D. MacFadden. Amsterdam: North-Holland.

Reid, N. (1995): "The Roles of Conditioning in Inference," *Statistical Science,* 10, 138-157.

——— (1996): "Likelihood and Bayesian approximation methods, in Bayesian Statistics 5 eds. J. M Bernardo, Jo. O. Berger, A. P Dawid, and A. F. M. Smith, Oxford Science Publications.

Solow, R.M. (1956): "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, LXX, 1956, 65-94.

Stuart, A., and J. K. Ord (1991): *Kendall's Advanced Theory of Statistics*, Vol. 2. New York: Oxford University Press.

Tibshirani, R., and L. Wasserman (1994): "Some Aspects of Reparametrization of Statistical Models," *The Canadian Journal of Statistics,* Vol. 22, 163-173.

Tierney, L. and J. B. Kadane (1986): "Accurate Approximations for Posterior Moments and Marginal Densities," Journal of the American Statistical Society, 81, 82-86.

Tierney, L., R. E. Kass and J. B. Kadane (1989): "Fully Exponential Laplace Approximations to Expectations and Variances of Nonpositive Functions," *Journal of the American Statistical Society*, 84, 710-716.

——— (1989): "Approximate Marginal Densities of Nonlinear Functions," *Biometrika*, 76, 425-433.

Trognon, A. (2000): “Panel Data Econometrics: A Succesful Past and a Promising Future,” Working Papaer Genes(INSEE).

Van der Vaart, A. W. (1998): *Asymptotic Statistics,* Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge.

Woutersen, T. M. (2000): “Consistent Estimation and Orthogonality,” UWO working paper.

Woutersen, T. M. (2001): “Robustness against Incidental Parameters and Mixing Distributions,” UWO working paper.

9 Tables

Table 1 Performances of integrated likelihood estimator for

the case without exogenous regressors comparative

to estimators found in Hahn, Hausman and Kuersteiner (2001)

(σ^2 is unknown; 5000 data sets)

T	n	ρ	$RMSE\ \hat{\rho}_{GMM}$	$RMSE\ \hat{\rho}_{BC2}$	$RMSE\ \hat{\rho}_{LIML}$	$RMSE\ \hat{\rho}_{I}$
5	100	0.10	0.08	0.08	0.082	0.056
10	100	0.10	0.05	0.05	0.045	0.035
5	500	0.10	0.04	0.04	0.036	0.025
10	500	0.10	0.02	0.02	0.020	0.016
5	100	0.30	0.10	0.10	0.099	0.061
10	100	0.30	0.05	0.05	0.050	0.036
5	500	0.30	0.04	0.04	0.044	0.027
10	500	0.30	0.02	0.02	0.023	0.016
5	100	0.50	0.13	0.13	0.130	0.067
10	100	0.50	0.06	0.06	0.058	0.037
5	500	0.50	0.06	0.06	0.057	0.031
10	500	0.50	0.03	0.03	0.026	0.016
5	100	0.80	0.32	0.34	0.327	0.089
10	100	0.80	0.14	0.11	0.109	0.045
5	500	0.80	0.13	0.13	0.127	0.036
10	500	0.80	0.05	0.04	0.044	0.018
5	100	0.90	0.55	0.78	0.604	0.099
10	100	0.90	0.25	0.23	0.229	0.057
5	500	0.90	0.28	0.30	0.277	0.038
10	500	0.90	0.10	0.08	0.080	0.023
5	100	0.95				0.111
10	100	0.95				0.059
5	500	0.95				0.042
10	500	0.95				0.026
5	100	0.99				0.118
10	100	0.99				0.058
5	500	0.99				0.044
10	500	0.99				0.028

T	n	ρ	$\%bias\ \hat{\rho}_{GMM}$	$\%bias\ \hat{\rho}_{BC2}$	$\%bias\ \hat{\rho}_{LIML}$	$\%bias\ \hat{\rho}_{I}$
5	100	0.10	-14.96	0.25	-3	2.66
10	100	0.10	-14.06	-0.77	-1	0.45
5	500	0.10	-3.68	-0.77	-1	-0.20
10	500	0.10	-3.15	-0.16	-1	-0.74
5	100	0.30	-8.86	-0.47	-3	1.04
10	100	0.30	-7.06	-0.66	-1	-0.25
5	500	0.30	-2.03	-0.16	-1	-0.37
10	500	0.30	-1.58	-0.10	0	-0.14
5	100	0.50	-10.05	-1.14	-3	0.38
10	100	0.50	-6.76	-0.93	-1	-0.15
5	500	0.50	-2.25	-0.15	-1	-0.29
10	500	0.50	-1.53	-0.11	0	-0.19
5	100	0.80	-27.65	-11.33	-15	0.80
10	100	0.80	-13.45	-4.55	-5	0.25
5	500	0.80	-6.98	-0.72	-3	-0.12
10	500	0.80	-3.48	-0.37	-1	-0.06
5	100	0.90	-50.22	-42.10	-41	1.36
10	100	0.90	-24.27	-15.82	-15	0.83
5	500	0.90	-20.50	-6.23	-10	-0.07
10	500	0.90	-8.74	-2.02	-2	-0.07
5	100	0.95				1.45
10	100	0.95				0.78
5	500	0.95				-0.05
10	500	0.95				0.02
5	100	0.99				1.69
10	100	0.99				0.44
5	500	0.99				0.15
10	500	0.99				0.22

The fixed effects α_i and the innovations ε_{it} are assumed to have independent standard normal distributions. Initial observations y_{i0} are assumed to be generated by the stationary distribution $N\left(\frac{\alpha_i}{1-\rho_0}, \frac{1}{1-\rho_0^2}\right)$.

Table 2 Performances of Integrated likelihood estimator for the case without exogenous regressors comparative to GMM and $BCML$ estimator of Hahn and Kuersteiner (2002)

(σ^2 is unknown; 5000 data sets)

T	n	ρ	$bias\ \hat{\rho}_{GMM}$	$bias\ \widehat{\hat{\rho}}$	$bias\ \hat{\rho}_I$	$RMSE\ \hat{\rho}_{GMM}$	$RMSE\ \widehat{\hat{\rho}}$	$RMSE\ \hat{\rho}_I$
5	100	0.0	−0.011	-0.039	−0.0004	0.074	0.065	0.054
5	100	0.3	−0.027	−0.069	0.003	0.099	0.089	0.061
5	100	0.6	−0.074	−0.115	0.002	0.160	0.129	0.070
5	100	0.9	−0.452	−0.178	0.012	0.552	0.187	0.099
5	200	0.0	−0.006	−0.041	−0.001	0.053	0.055	0.038
5	200	0.3	−0.014	−0.071	0.001	0.070	0.081	0.042
5	200	0.6	−0.038	−0.116	0.003	0.111	0.124	0.048
5	200	0.9	−0.337	−0.178	0.006	0.443	0.183	0.067
10	100	0.0	−0.011	−0.010	−0.001	0.044	0.036	0.035
10	100	0.3	−0.021	−0.019	−0.0007	0.053	0.040	0.036
10	100	0.6	−0.045	−0.038	0.001	0.075	0.051	0.037
10	100	0.9	−0.218	−0.079	0.007	0.248	0.085	0.058
10	200	0.0	−0.006	−0.011	−0.0007	0.031	0.027	0.025
10	200	0.3	−0.011	−0.019	−0.001	0.038	0.032	0.026
10	200	0.6	−0.025	−0.037	−0.0001	0.051	0.045	0.027
10	200	0.9	−0.152	−0.079	0.004	0.181	0.082	0.041
20	100	0.0	−0.011	−0.003	−0.000	0.029	0.024	0.023
20	100	0.3	−0.017	−0.005	0.000	0.033	0.024	0.023
20	100	0.6	−0.029	−0.011	0.000	0.042	0.024	0.022
20	100	0.9	−0.100	−0.032	0.0005	0.109	0.037	0.026
20	200	0.0	−0.006	−0.003	−0.0002	0.020	0.017	0.017
20	200	0.3	−0.009	−0.005	−0.0002	0.022	0.017	0.017
20	200	0.6	−0.016	−0.010	−0.0004	0.027	0.018	0.015
20	200	0.9	−0.065	−0.031	0.0008	0.074	0.034	0.018

The fixed effects α_i and the innovations ε_{it} are assumed to have independent standard normal distributions. Initial observations y_{i0} are assumed to be generated by the stationary distribution $N\left(\frac{\alpha_i}{1-\rho_0}, \frac{1}{1-\rho_0^2}\right)$.

Table 3. Performances of Integrated likelihood estimator for high values of ρ and $T = 5$

$N = 100$		$\widehat{\rho}_{LIML,1}$	$\widehat{\rho}_{I2SLS,LD}$	$\widehat{\rho}_{CUE,LD}$	$\widehat{\rho}_{I}$
$\rho = 0.75$	*Actual mean % Bias*	1.297	5.533	11.553	1.158
$\rho = 0.75$	*Actual median % Bias*	-3.087	1.381	7.470	0.475
$\rho = 0.75$	$RMSE$	0.181	0.176	0.213	0.084
$\rho = 0.80$	*Actual mean % Bias*	-0.112	4.304	10.413	0.802
$\rho = 0.80$	*Actual median % Bias*	-5.725	1.457	8.651	-0.112
$\rho = 0.80$	$RMSE$	0.213	0.173	0.205	0.089
$\rho = 0.85$	*Actual mean % Bias*	-3.899	1.966	7.983	0.995
$\rho = 0.85$	*Actual median % Bias*	-10.117	0.065	7.558	-0.022
$\rho = 0.85$	$RMSE$	0.233	0.160	0.194	0.093
$\rho = 0.90$	*Actual mean % Bias*	-9.757	-0.771	6.138	1.363
$\rho = 0.90$	*Actual median % Bias*	-15.389	-2.346	6.114	0.288
$\rho = 0.90$	$RMSE$	0.246	0.153	0.180	0.099
$\rho = 0.95$	*Actual mean % Bias*	-15.203	-3.367	3.124	1.455
$\rho = 0.95$	*Actual median % Bias*	-19.637	-4.776	3.136	-0.045
$\rho = 0.95$	$RMSE$	0.252	0.149	0.165	0.111

Table 4 Integrated likelihood estimator for the exogenous regressors case

using Lancaster's approximation (1997)

(σ^2 is unknown; 5000 data sets, σ_ξ^2 is the variance of the exogenous variable generation process)

	$BIAS\ \rho$	$BIAS\ \beta$	$RMSE\ \rho$	$RMSE\ \beta$	T	ρ	β	α	σ_ξ
I	0.065	-0.044	0.079	0.156	6	0.0	1.0	0.80	0.85
II	0.034	-0.018	0.053	0.135	6	0.4	0.6	0.80	0.88
III	0.006	-0.001	0.033	0.174	6	0.8	0.2	0.80	0.40
IV	0.050	-0.042	0.064	0.063	6	0.0	1.0	0.99	0.20
V	0.027	-0.019	0.046	0.045	6	0.4	0.6	0.99	0.19
VI	0.005	-0.002	0.032	0.033	6	0.8	0.2	0.99	0.07
	$BIAS\ \rho$	$BIAS\ \beta$	$RMSE\ \rho$	$RMSE\ \beta$	T	ρ	β	α	σ_ξ
VII	0.028	-0.010	0.072	0.186	3	0.4	0.6	0.80	0.88
$VIII$	0.027	-0.007	0.071	0.136	3	0.4	0.6	0.80	1.84
XI	0.020	-0.008	0.062	0.051	3	0.4	0.6	0.99	0.19
XII	0.023	-0.010	0.063	0.052	3	0.4	0.6	0.99	0.40

Kieviet Results for LSDVc estimator in the exogenous regressors case

	$BIAS\ \rho$	$BIAS\ \beta$	$RMSE\ \rho$	$RMSE\ \beta$	T	ρ	β	α	σ_ξ
I	−0.019	−0.018	0.043	0.057	6	0.0	1.0	0.80	0.85
II	−0.038	−0.002	0.059	0.052	6	0.4	0.6	0.80	0.88
III	−0.125	−0.011	0.135	0.114	6	0.8	0.2	0.80	0.40
IV	−0.017	0.005	0.050	0.217	6	0.0	1.0	0.99	0.20
V	−0.042	0.019	0.067	0.231	6	0.4	0.6	0.99	0.19
VI	−0.127	0.050	0.136	0.643	6	0.8	0.2	0.99	0.07
	$BIAS\ \rho$	$BIAS\ \beta$	$RMSE\ \rho$	$RMSE\ \beta$	T	ρ	β	α	σ_ξ
VII	−0.205	0.004	0.220	0.093	3	0.4	0.6	0.80	0.88
$VIII$	−0.033	0.061	0.077	0.077	3	0.4	0.6	0.80	1.84
XI	−0.249	0.055	0.263	0.433	3	0.4	0.6	0.99	0.19
XII	−0.248	0.046	0.263	0.212	3	0.4	0.6	0.99	0.40

Table 5. Integrated likelihood estimator for the exogenous regressors case

(σ^2 is unknown; 5000 data sets)

	$BIAS\ \rho$	$BIAS\ \beta$	$RMSE\ \rho$	$RMSE\ \beta$	T	ρ	β	α	σ_ξ
I	−0.0003	0.0004	0.0288	0.0288	6	0.0	1.0	0.80	0.85
II	0.0002	0.0006	0.0289	0.0283	6	0.4	0.6	0.80	0.88
III	0.0002	0.0005	0.0290	0.0288	6	0.8	0.2	0.80	0.40
IV	0.0004	−0.0004	0.0289	0.0290	6	0.0	1.0	0.99	0.20
V	−0.0001	0.0004	0.0287	0.0289	6	0.4	0.6	0.99	0.19
VI	−0.0004	−0.0007	0.0290	0.0289	6	0.8	0.2	0.99	0.07
	$BIAS\ \rho$	$BIAS\ \beta$	$RMSE\ \rho$	$RMSE\ \beta$	T	ρ	β	α	σ_ξ
VII	−0.0009	0.0010	0.0287	0.0287	3	0.4	0.6	0.80	0.88
$VIII$	−0.0001	−0.0010	0.0290	0.0288	3	0.4	0.6	0.80	1.84
XI	−0.0002	−0.0008	0.0287	0.0289	3	0.4	0.6	0.99	0.19
XII	−0.0020	0.0003	0.0283	0.0289	3	0.4	0.6	0.99	0.40